HEAT INVARIANTS OF RIEMANNIAN MANIFOLDS

BY

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ABSTRACT

We calculate heat invariants of arbitrary Riemannian manifolds without boundary. Every heat invariant is expressed in terms of powers of the Laplacian and the distance function. Our approach is based on a multi-dimensional generalization of the Agmon–Kannai method. An application to computation of the Korteweg–de Vries hierarchy is also presented.

1. Introduction and main results

1.1. HEAT INVARIANTS. Let M be a d-dimensional Riemannian manifold without boundary with a metric (g_{ij}), and Δ be the Laplace–Beltrami operator (or simply the **Laplacian**) on M. In local coordinates $(x_1, \ldots, x_d)$ the Laplacian is given by

$$(1.1.1) \qquad \Delta f = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{d} \frac{\partial(\sqrt{g}g^{ij}(\partial f/\partial x_i))}{\partial x^j},$$

where $g = \det(g_{ij})$, and (g^{ij}) denotes the inverse of the matrix (g_{ij}).

The **heat kernel** $K(t, x, y)$ is the fundamental solution of the heat equation

$$\left(\frac{\partial}{\partial t} + \Delta\right)f = 0.$$

The function $K(t, x, y)$ is analytic in $t > 0$ and C^∞ in x and y, and has the following asymptotic expansion on the diagonal as $t \to 0+$ (see [G1]):

$$K(t, x, x) \sim \sum_{n=0}^{\infty} a_n(x) t^{n-d/2}.$$

It is called the Minakshisundaram-Pleijel asymptotic expansion (see [MP]). The coefficients $a_n(x)$ are (local) **heat invariants** of the manifold M. They are homogeneous polynomials of degree $2n$ in the derivatives of the Riemannian metric $\{g^{ij}\}$ at the point x ([G2]). Integrating $a_n(x)$ over the manifold one gets the coefficients a_n of the expansion for the trace of the heat operator $e^{-t\Delta}$:

$$(1.1.2) \qquad \sum_i e^{-t\lambda_i} \sim \sum_{n=0}^{\infty} \left(\int_M a_n(x) \sqrt{g}\, dx \right) t^{n-d/2} \sim \sum_{n=0}^{\infty} a_n t^{n-d/2}.$$

Computation of heat invariants is a well-known problem in spectral geometry (see [BGM], [Be], [G1], [Ch], [G3], [Ro]) which has various applications ([F], [P2]). The first method for derivation of heat kernel asymptotics is due to Seeley ([Se]). This method was developed later by Gilkey (see Theorem 1.3 in [G1]) who presented a way to get recursive formulas for the heat invariants.

However, explicit formulas for $a_n(x)$ in arbitrary dimension existed only for $n \leq 5$ ([MS], [Sa], [Av], [vdV]). The reason for this is the combinatorial complexity of $a_n(x)$ which is increasing very rapidly with the growth of n. For the higher heat invariants only partial information is known ([BGØ], [OPS]). Let us also mention interesting recursive formulas for $a_n(x)$ obtained in [Xu].

In this paper we represent all heat invariants $a_n(x)$ of an arbitrary Riemannian manifold without boundary in terms of powers of the Laplacian and the distance function. This is different from the classical way of expressing heat invariants — using the curvature tensor and its derivatives. Expanding the distance function by means of curvatures is a highly non-trivial computational problem (to say nothing about applying powers of the Laplacian to this function). We refer to [Gra], [GraV] for results on such expansions.

Let us mention, however, that heat invariants were first given by a recursive system of differential equations involving exactly the Laplacian and the distance function (see [MP]).

1.2. MAIN RESULT. Given a point $x \in M$ denote by $\rho_x \colon M \to \mathbb{R}$ the corresponding distance function: for every $y \in M$ the distance between the points y and x is $\rho_x(y)$.

THEOREM 1.2.1: *Heat invariants $a_n(x)$ are equal to*

$$a_n(x) = (4\pi)^{-d/2}(-1)^n \sum_{j=0}^{3n} \binom{3n + \frac{d}{2}}{j + \frac{d}{2}} \frac{1}{4^j\, j!\,(j+n)!}\, \Delta^{j+n}(\rho_x(y)^{2j})\big|_{y=x}\,.$$

The binomial coefficients for d odd are defined by (4.1.2).

1.1. STRUCTURE OF THE PAPER.

In [P1], [P2] we have developed a method for computation of heat invariants based on the Agmon–Kannai asymptotic expansion of resolvent kernels of elliptic operators ([AK]). In [P1] this method is used to obtain explicit formulas for the heat invariants of 2-dimensional Riemannian mainfolds, and in [P2] for computation of the Korteweg–de Vries (KdV) hierarchy via heat kernel coefficients of the 1-dimensional Schrödinger operator. In this paper we present a multi-dimensional generalization of the Agmon–Kannai method which is described in section 2.3. In section 3.1 we apply it to get formulas for the heat invariants in normal coordinates. It turns out that combinatorial coefficients in these formulas can be substantially simplified, which is done in section 4.1. In section 4.2 we present $a_n(x)$ in a completely invariant form and prove Theorem 1.2.1. The main result allows one to simplify the formulas for the KdV hierarchy obtained in [P2]. This is shown in sections 5.1 and 5.2.

2. Asymptotics of derivatives of the resolvent

2.1. A MODIFICATION OF AGMON–KANNAI EXPANSION.

The original Agmon–Kannai theorem ([AK]) deals with asymptotic behaviour of resolvent kernels of elliptic operators. In [P1] we have obtained a concise reformulation of this theorem which is suitable for computation of heat invariants. We start with some notations.

Let H be a a self-adjoint elliptic differential operator of order p on a Riemannian manifold (M, g_{ij}) of dimension $d < p$ and let H_0 be the operator obtained by freezing the coefficients of the principal part H' of the operator H at some point $x \in M$: $H_0 = H'(x)$. Denote by $R_\lambda(x,y)$ the kernel of the resolvent $R_\lambda = (H - \lambda)^{-1}$, and by $F_\lambda(x,y)$ the kernel of $F_\lambda = (H_0 - \lambda)^{-1}$.

THEOREM 2.1.1 ([P1]): *The resolvent kernel $R_\lambda(x,y)$ has the following asymptotic representation on the diagonal as $\lambda \to \infty$:*

$$(2.1.2) \qquad R_\lambda(x,x) \sim \frac{1}{\sqrt{g}} \sum_{m=0}^{\infty} X_m F_\lambda^{m+1}(x,x),$$

where the operators X_m are defined by

$$(2.1.3) \qquad X_m = \sum_{k=0}^{m}(-1)^k \binom{m}{k} H^k H_0^{m-k}, \quad m \geq 0.$$

2.2. DERIVATIVES OF THE RESOLVENT. The main obstruction in using Theorem 2.1.1 directly for computation of heat invariants of a d-dimensional Riemannian manifold is the condition $d < p$, where $p = 2$ is the order of the Laplacian. In [P1] we avoid this difficulty for 2-dimensional manifolds taking the difference of resolvents. However, in the general case one should consider derivatives of the resolvent kernel (cf. [AvB]).

LEMMA 2.2.1: *The following asymptotic expansion on the diagonal holds for the derivatives of the resolvent kernel of the Laplacian on a d-dimensional Riemannian manifold M:*

$$(2.2.2) \quad \frac{d^s}{d\lambda^s} R_\lambda(x,x) \sim \sum_{n=0}^{\infty} \Gamma(s + n - d/2 + 1) a_n(x)(-\lambda)^{d/2 - s - n - 1}, \quad s \geq d/2,$$

where $a_n(x)$ are heat invariants of the manifold M.

Proof: Let $\operatorname{Re} \lambda < 0$. We have (formally)

$$\int_0^\infty e^{-t(\Delta - \lambda)} dt = \frac{1}{\Delta - \lambda}.$$

Differentiating R_λ s times with respect to λ we get a self-adjoint operator from $L^2(M)$ into the Sobolev space $\mathrm{H}^{2s+2}(M)$. Since $2s + 2 > \dim M$ this operator has a continuous kernel (see [AK]). Taking into account (1.1.2) we formally have

$$(2.2.3) \qquad \frac{d^s}{d\lambda^s}\left(\frac{1}{\Delta - \lambda}\right) = \int_0^\infty t^s e^{-t(\Delta - \lambda)} dt \sim \sum_{n=0}^{\infty} a_n \int_0^\infty t^{s+n-d/2} e^{\lambda t} dt.$$

The asymptotic expansion in (2.2.3) is obviously valid if we integrate over a finite interval $[0, T]$. In order to show that it remains true in our case as well we need an additional argument.

Indeed, it is well-known (for example, see [Da]) that

$$|e^{-t\Delta}| \leq ct^{-d/2}.$$

Therefore we have

$$\left| \int_0^\infty t^s e^{-t(\Delta - \lambda)} dt - \int_0^T t^s e^{-t(\Delta - \lambda)} dt \right| \leq \int_T^\infty ct^{s-d/2} e^{\lambda t} dt.$$

Let us estimate the second integral. We have

$$\int_T^\infty t^{s-d/2}e^{\lambda t}dt \le e^{-\varepsilon T}\int_T^\infty t^{s-d/2}e^{(\lambda+\varepsilon)t}dt \le e^{-\varepsilon T}\frac{\Gamma(s-d/2+1)}{(\lambda+\varepsilon)^{s-d/2+1}}.$$

Take $\varepsilon = \sqrt{|\lambda|}$. Then for $T = 1$ this is $O(e^{-\sqrt{|\lambda|}})$ and therefore the term

$$\int_T^\infty t^{s-d/2}e^{\lambda t}dt$$

is negligible. This proves the asymptotic expansion in (2.2.3).

The right-hand side of (2.2.3) is equal to

$$\sum_{n=0}^\infty \frac{a_n}{(-\lambda)^{s+n+1-d/2}}\int_0^\infty u^{s+n-d/2}e^{-u}du = \sum_{n=0}^\infty \frac{\Gamma(s+n+1-d/2)a_n}{(-\lambda)^{s+n+1-d/2}},$$

and this completes the proof of the lemma. ∎

2.3. AGMON–KANNAI EXPANSION FOR DERIVATIVES OF THE RESOLVENT. In the notations of Theorem 2.1.1 let $H = \Delta$ be the Laplacian on a d-dimensional Riemannian manifold M, and Δ_0 be the operator obtained from the principal part of the Laplacian by freezing its coefficients at a certain point $x \in M$. As before $R_\lambda = (\Delta - \lambda)^{-1}$ and $F_\lambda = (\Delta_0 - \lambda)^{-1}$.

THEOREM 2.3.1: *The following asymptotic expansion on the diagonal holds for the derivatives of the resolvent kernel of the Laplacian on a d-dimensional Riemannian manifold M:*

$$(2.3.2)\qquad \frac{d^s}{d\lambda^s}R_\lambda(x,x) \sim \frac{1}{\sqrt{g}}\sum_{m=0}^\infty \frac{(m+s)!}{m!}X_m F_\lambda^{m+s+1}, \quad s \ge d/2.$$

Proof: Formally we have

$$\frac{d}{d\lambda}F_\lambda = \frac{d}{d\lambda}\left(\frac{1}{\Delta_0 - \lambda}\right) = \frac{1}{(\Delta_0 - \lambda)^2} = F_\lambda^2.$$

This implies

$$\frac{d^s}{d\lambda^s}F_\lambda^{m+1} = \frac{(m+s)!}{m!}F_\lambda^{m+s+1}.$$

Together with (2.1.2) this completes the proof of the theorem. ∎

Let us introduce the standard multi-index notations (see [Hö]): if $\alpha = (\alpha_1,\ldots,\alpha_d)$ is a multi-index, then $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\alpha! = \alpha_1!\cdots\alpha_d!$. For any vector $x = (x_1,\ldots,x_d)$ we denote $x^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ and

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$

We note that (2.3.2) and (2.1.2) are in fact asymptotic expansions in powers of $-\lambda$ as well as (2.2.2). This is due to the following formula (see [AK]):

$$(2.3.3) \qquad \frac{\partial^\gamma}{\partial x^\gamma} F_\lambda^{m+s+1}(x, x) =$$

$$(-\lambda)^{\frac{(d+|\gamma|)}{2}-m-s-1} \frac{(-1)^{|\gamma|/2}}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\xi^\gamma \, d\xi}{(\Delta_0(\xi)^2 + 1)^{m+s+1}},$$

where $\Delta_0(\xi)$ denotes the symbol of the operator Δ_0, $\gamma = (\gamma_1, \ldots, \gamma_d)$ is a multi-index and $\xi = (\xi_1, \ldots, \xi_d)$.

3. Heat invariants in normal coordinates

3.1. COMPUTATION OF HEAT INVARIANTS. Let $(x_1, \ldots, x_d)$ be local coordinates on the Riemannian manifold M such that the Riemannian metric at the origin $x = (0, \ldots, 0) \in M$ (in the sequel we simply write $x = 0$) is Euclidean: $g_{ij}|_{x=0} = \delta_{ij}$. For convenience we may consider normal coordinates on M centered at the point $x = 0$ (see [GKM]).

THEOREM 3.1.1: *Let M be a d-dimensional Riemannian manifold without boundary and $(x_1, \ldots, x_d)$ be normal coordinates on M centered at the point $x = 0$. Then the heat invariants $a_n(x)$ at the point $x = 0$ are equal to*

$$(3.1.2) \quad a_n(0) =$$

$$(4\pi)^{-d/2}(-1)^n \sum_{m=n}^{4n} \sum_{k=n}^{m} \frac{1}{k! \, 2^{2m-2n}} \sum_{|\alpha|=m-k} \sum_{|\beta|=k-n} \frac{(2\alpha+2\beta)!}{\alpha!(\alpha+\beta)!(2\beta)!} \, \Delta^k(x^{2\beta})\big|_{x=0},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ are multi-indices.

Proof of Theorem 3.1.1: Since $(x_1, \ldots, x_d)$ are normal coordinates centered at $x = 0$, the principal part of the Laplacian at this point coincides with the Euclidean Laplacian, i.e.

$$(3.1.3) \qquad\qquad \Delta_0 = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_d^2}.$$

Due to (2.2.2), in order to compute the coefficient $a_n(x)$ we have to collect all terms in the expansion (2.3.2) containing $(-\lambda)^{d/2-s-n-1}$. From (2.3.3) we have

$$\frac{d + |\gamma|}{2} - m - s - 1 = \frac{d}{2} - s - n - 1,$$

which implies $|\gamma| = 2m - 2n$ and in particular $m \geq n$. As was shown in [P1], estimates on the orders of operators X_m (namely, Lemma 3.1 and Theorem 5.1 in [AK]) imply that $m \leq 4n$.

Note that due to (3.1.3) all indices $\gamma_1, \ldots, \gamma_2$ should be even since otherwise the integral in (2.3.3) will vanish. Setting $\gamma = 2\mu = (2\mu_1, \ldots, 2\mu_d)$ and taking into account that $|\mu| = m - n$ we compute this integral (see [GR]):

$$\int\limits_{\mathbb{R}^d} \frac{\xi^{2\mu} \, d\xi}{(\xi^2 + 1)^{m+s+1}} = \frac{\Gamma(\mu_1 + \frac{1}{2})\Gamma(\mu_2 + \frac{1}{2}) \cdots \Gamma(\mu_d + \frac{1}{2})\Gamma(s + n + 1 - \frac{d}{2})}{(m+s)!}.$$

Substituting this into (2.3.3) and further on into (2.3.2) we obtain, due to (2.2.2):

$$a_n(x) =$$
$$\sum_{m=n}^{4n} \sum_{k=0}^{m} \frac{(-1)^{k+m-n}}{m!(2\pi)^d} \binom{m}{k} \Delta^k \Delta_0^{m-k} \left(\sum_{|\mu|=m-n} \frac{x^{2\mu}}{(2\mu)!} \prod_{i=1}^{d} \Gamma\left(\mu_i + \frac{1}{2}\right) \right)\Bigg|_{x=0}.$$

Note that s has cancelled out, as one would expect since heat invariants do not depend on s!

Now let us simplify this formula. First notice that

$$(3.1.4) \qquad \Delta_0^{m-k} = (-1)^{m-k} \sum_{|\beta|=m-k} \frac{(m-k)!}{\beta!} \frac{\partial^{2\beta}}{\partial x^{2\beta}},$$

where $\beta = (\beta_1, \ldots, \beta_d)$. Using the well-known representation of the Γ-function

$$(3.1.5) \qquad \Gamma(k + 1/2) = \frac{\sqrt{\pi}(2k)!}{4^k k!}$$

we also obtain

$$(3.1.6) \qquad \prod_{i=1}^{d} \Gamma\left(\mu_i + \frac{1}{2}\right) = \frac{\pi^{d/2}}{2^{2m-2n}} \frac{(2\mu)!}{\mu!}.$$

Let us substitute (3.1.4) and (3.1.6) into the above formula for the $a_n(x)$ and apply Δ_0^{m-k} to $x^{2\mu}$. Introducing the new summation multi-index $\alpha = \mu - \beta$ and noticing that all terms for $k < n$ vanish, we finally obtain

$$a_n(0) =$$
$$(4\pi)^{-d/2}(-1)^n \sum_{m=n}^{4n} \sum_{k=n}^{m} \frac{1}{k! \, 2^{2m-2n}} \sum_{|\alpha|=m-k} \sum_{|\beta|=k-n} \frac{(2\alpha + 2\beta)!}{\alpha!(\alpha + \beta)!(2\beta)!} \ \Delta^k(x^{2\beta})\big|_{x=0}.$$

This completes the proof of the theorem. ∎

3.2. Remarks: The proof of Theorem 3.1.1 is similar to the proofs of the main theorems in [P1] and [P2]. One may check that in the particular cases of the 2-dimensional Laplacian and the 1-dimensional Schrödinger operator, Theorem 3.1.1 agrees with the results obtained in [P1] and [P2].

4. Invariance and combinatorial identities

4.1. COMBINATORIAL IDENTITIES.   Let us rewrite (3.1.2) in the following way:

$$(4\pi)^{-d/2}(-4)^n \times$$

$$\sum_{k=n}^{4n} \frac{1}{k!} \Delta^k \left(\sum_{m=k}^{4n} \frac{1}{4^m} \sum_{|\beta|=k-n} \left(\sum_{|\alpha|=m-k} \frac{(2\alpha+2\beta)!\beta!}{\alpha!(\alpha+\beta)!(2\beta)!} \right) \frac{1}{\beta!} x^{2\beta} \right) \Bigg|_{x=0}.$$

Observe that due to the multinomial theorem

$$(4.1.1) \qquad \sum_{|\beta|=k-n} \frac{1}{\beta!} x^{2\beta} = \frac{1}{(k-n)!}(x_1^2 + \cdots + x_d^2)^{k-n}.$$

Let us recall the following generalization of the binomial coefficients (see [Er]). For real $z \in \mathbb{R}$ and $a \in \mathbb{N}$ set

$$(4.1.2) \qquad \binom{z}{a} = \binom{z}{z-a} = \frac{\Gamma(z+1)}{\Gamma(a+1)\Gamma(z-a+1)} = \frac{z(z-1)\cdots(z-a+1)}{a!}.$$

We also set $\binom{z}{0} = \binom{z}{z} = 1$.

Let us proceed with the following simple combinatorial formula.

LEMMA 4.1.3:

$$(4.1.4) \qquad \sum_{a=0}^{u} \binom{z+a}{a}\binom{w+u-a}{u-a} = \binom{z+w+u+1}{z+w+1}.$$

Proof: Using the method of generating functions (see [Rio]) we have

$$\sum_{a=0}^{\infty} \binom{z+a}{a} q^{2a} = \frac{1}{(1-q^2)^{z+1}},$$

which implies

$$\sum_{u=0}^{\infty} \sum_{a_1+a_2=u} \binom{z+a_1}{a_1}\binom{w+a_2}{a_2} q^{2u} = \frac{1}{(1-q^2)^{z+w+2}} = \sum_{u=0}^{\infty} \binom{z+w+u+1}{z+w+1} q^{2u}.$$

This completes the proof of the lemma.   ∎

Now we can prove our main combinatorial identity.

THEOREM 4.1.5: *Let α and β be multi-indices of dimension d and let $|\beta| = v$. Then*

$$(4.1.6) \qquad \sum_{|\alpha|=u} \frac{(2\alpha + 2\beta)!\beta!}{\alpha!(\alpha + \beta)!(2\beta)!} = 4^u \binom{u + v - 1 + d/2}{u}.$$

Proof: We proceed by induction over d. For $d = 1$ we have, due to (3.1.5),

$$\frac{(2u + 2v)!v!}{u!(u+v)!(2v)!} = \frac{4^{u+v}\Gamma(u+v+1/2)\sqrt{\pi}}{4^v\Gamma(v+1/2)u!\sqrt{\pi}} = 4^u \binom{u + v - 1/2}{u},$$

and hence (4.1.6) is valid.

Suppose we have proved formula (4.1.6) in all dimensions less than some $d > 1$. Let us prove it in the dimension d. Denote $\alpha_1 = a$ and $\beta_1 = b$. By induction we may rewrite the sum in (4.1.6) as

$$(4.1.7) \qquad \sum_{|\alpha|=u} \frac{(2\alpha + 2\beta)!\beta!}{\alpha!(\alpha + \beta)!(2\beta)!} = \sum_{a=0}^{u} \frac{(2a + 2b)!b!}{a!(a + b)!(2b)!} \binom{u - a - 1 + l}{u - a} 4^{u-a},$$

where $l = v - b + (d - 1)/2$. On the other hand,

$$\frac{1}{4^a} \frac{(2a + 2b)!b!}{a!(a + b)!(2b)!} = \binom{a + b - 1/2}{a}$$

and hence (4.1.7) is equal to

$$4^u \sum_{a=0}^{u} \binom{a + b - 1/2}{a} \binom{u - a - 1 + l}{u - a}.$$

By Lemma 4.1.3 this equals

$$4^u \binom{u + b + l - 1/2}{u} = 4^u \binom{u + v - 1 + d/2}{u},$$

which completes the proof of the theorem. ∎

4.2. Proof of Theorem 1.2.1: Set $u = m - k$ and $v = k - n$. Combining Theorem 3.1.1, Theorem 4.1.5 and formula (4.1.1) we obtain the following reformulation of (3.1.2):

$$(4.2.1) \quad a_n(0) = (4\pi)^{-d/2}(-4)^n \sum_{k=n}^{4n} \left(\sum_{m=k}^{4n} \binom{m + \frac{d}{2} - n - 1}{m - k} \right) \frac{\Delta^k(|x|^{2k-2n})|_{x=0}}{k!\, 4^k},$$

where $|x|^2 = x_1^2 + \cdots + x_d^2$. Denote $i = m - k$ and $j = k - n$. By Lemma 4.1.3 the inner sum may be rewritten as

$$\sum_{i=0}^{4n-k} \binom{i + d/2 + k - n - 1}{i} = \binom{3n + d/2}{j + d/2}.$$

Therefore (4.2.1) is equal to

$$(4.2.2) \qquad a_n(0) = (4\pi)^{-d/2}(-1)^n \sum_{j=0}^{3n} \binom{3n + d/2}{j + d/2} \frac{\Delta^{j+n}(|x|^{2j})|_{x=0}}{4^j \, j! \, (j+n)!}.$$

Consider the function $\rho_x(y)^2$, which is the square of the distance between the points x and y. In normal coordinates centered at the point $x = 0$ it is given locally by

$$\rho_x(y)^2 = \sum_{i,j=1}^{d} g_{ij}(0) \, y_i y_j = y_1^2 + \cdots + y_d^2 = |y|^2$$

where $y = (y_1, \ldots, y_d)$ (see [Du], p. 94). Therefore we may rewrite formula (4.2.2) in an invariant form, namely

$$a_n(x) = (4\pi)^{-d/2}(-1)^n \sum_{j=0}^{3n} \binom{3n + d/2}{j + d/2} \frac{\Delta_y^{j+n}(\rho_x(y)^{2j})|_{y=x}}{4^j \, j! \, (j+n)!},$$

where the subscript of the Laplacian means that the operator is acting on functions in the y-variable. This completes the proof of Theorem 1.2.1. ∎

5. Application to computation of the Korteweg–de Vries hierarchy

5.1. SCHRÖDINGER OPERATOR ASYMPTOTICS AND KDV HIERARCHY.   In [P2] we have applied the Agmon–Kannai method to computation of the Korteweg–de Vries hierarchy (see [NMPZ]). Let us briefly recall the setting of the problem.

Consider the 1-dimensional Schrödinger operator

$$L = \frac{\partial^2}{\partial x^2} + U(x).$$

Its heat kernel $H(t, x, y)$ is the fundamental solution of the heat equation

$$\left(\frac{\partial}{\partial t} - L \right) f = 0.$$

It has the following asymptotic representation on the diagonal as $t \to 0+$:

$$H(t, x, x) \sim \frac{1}{\sqrt{4\pi t}} \sum_{n=0}^{\infty} h_n[U] t^n,$$

where $h_n[U]$ are some polynomials in $U(x)$ and its derivatives.

The KdV hierarchy is defined by (see [AvSc])

(5.1.1) $$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x} G_n[U],$$

where

$$G_n[U] = \frac{(2n)!}{2 \cdot n!} h_n[U], \quad n \in \mathbb{N}.$$

Set $U_0 = U$, $U_n = \partial^n U/\partial x^n$, $n \in \mathbb{N}$, where U_n, $n \geq 0$ are formal variables. The sequence of polynomials $G_n[U] = G_n[U_0, U_1, U_2, \ldots]$ starts with (see [AvSc])

$$G_1[U] = U_0, \quad G_2[U] = U_2 + 3U_0^2, \quad G_3[u] = U_4 + 10U_0 U_2 + 5U_1^2 + 10U_0^3, \ldots.$$

In particular, substituting $G_2[U]$ into (5.1.1) we obtain the familiar Korteweg–de Vries equation (see [NMPZ])

$$\frac{\partial U}{\partial t} = \frac{\partial^3 U}{\partial x^3} + 6U \frac{\partial U}{\partial x}.$$

5.2. COMPUTATION OF THE KDV HIERARCHY. In [P2] we have presented explicit formulas for the KdV hierarchy (we refer to [P2] for the history of this question). Theorem 1.2.1 allows us to simplify the results of [P2].

THEOREM 5.2.1: *The KdV hierarchy is given by*

$$G_n[U] = \frac{(2n)!}{2 \cdot n!} \sum_{j=0}^{n} \binom{n + \frac{1}{2}}{j + \frac{1}{2}} \frac{(-1)^j}{4^j \, j! \, (j+n)!} P_{nj}[U],$$

where the polynomial $P_{nj}[U]$ is obtained from $L^{j+n}(x^{2j})\big|_{x=0}$ by a formal change of variables: $U_i(0) \to U_i$, $i = 0, \ldots, 2n + 2j - 2$.

This expression can be completely expanded due to a formula for the powers of the Schrödinger operator ([Rid]).

THEOREM 5.2.2: *The polynomials $G_n[U]$, $n \in \mathbb{N}$ are equal to*

$$G_n[U] = \frac{(2n)!}{2 \cdot n!} \sum_{j=0}^{n} \binom{n + \frac{1}{2}}{j + \frac{1}{2}} \frac{(-1)^j (2j)!}{4^j \, j! \, (j+n)!} \sum_{p=1}^{j+n} \sum_{\substack{k_1, \ldots, k_p \\ k_1 + \cdots + k_p = 2(n-p)}} C_{k_1, \ldots, k_p} U_{k_1} \cdots U_{k_p},$$

where

$$C_{k_1,\ldots,k_p} =$$

$$\sum_{\substack{0 \le l_0 \le l_1 \le \cdots \le l_{p-1} = j+n-p \\ 2l_i \ge k_1 + \cdots + k_{i+1},\ i=0,\ldots,p-1.}} \binom{2l_0}{k_1}\binom{2l_1 - k_1}{k_2} \cdots \binom{2l_{p-1} - k_1 - \cdots - k_{p-1}}{k_p}.$$

Remark: Theorem 5.2.1 was checked using Mathematica ([Wo]) and for $1 \le n \le 5$ the results agreed with those already known (cf. [GD]).

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